MATH 2050C Lecture 4 (Jan 20)

[Quiz 1 in-class on Jan 25.]

Goal: $\mathbb{R}$ is a complete ordered field.

Completeness Property: Every $\phi \neq S \subseteq \mathbb{R}$ which is bounded above must have a supremum in $\mathbb{R}$. [Note: $\mathbb{Q}$ fails this!]

Last time, we proved:

Prop: $u = \sup S$ iff

① $s \leq u$, $\forall s \in S$ (i.e. u is an upper bd)

② $\forall \varepsilon > 0$, $\exists s' \in S$ s.t. $u - \varepsilon < s'$ (i.e. u is the smallest upper bd.)

Similarly, for infimum, we have:

Prop: $u = \inf S$ iff

① $s \geq u$, $\forall s \in S$ (i.e. u is an lower bd)

② $\forall \varepsilon > 0$, $\exists s' \in S$ s.t. $u + \varepsilon > s'$ (i.e. u is the greatest lower bd.)

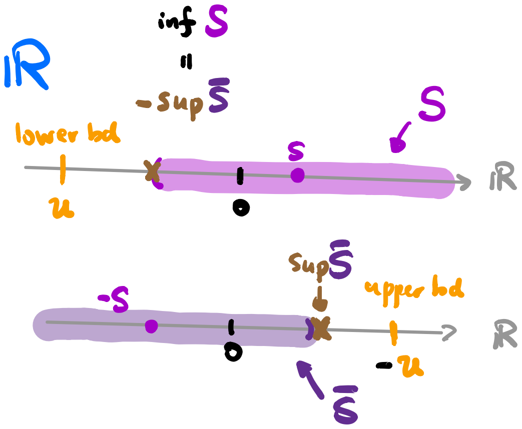
Q: What about the existence of infimum?

A: It follows from the completeness property.

Prop: Every $\phi \neq S \subseteq \mathbb{R}$ that is bounded below must have an infimum in $\mathbb{R}$.

Proof: Given $\phi \neq S \subseteq \mathbb{R}$, consider the subset:

$$\phi \neq \bar{S} := \{ -s \mid s \in S \} \subseteq \mathbb{R}$$



Claim: $\bar{S}$ is bdd above.

Pf of Claim:

Since S is bdd below, i.e.

$\exists$ some lower bd. u of S

$\Leftrightarrow \quad u \leq s \quad \forall s \in S$

$\Rightarrow \quad -u \geq -s \quad \forall s \in S$

$\Rightarrow \quad -u$ is an upper bd for $\bar{S}$ ie. $\bar{S}$ is bdd above.

By Completeness Property, $\sup \bar{S}$ exists in $\mathbb{R}$.

Claim: $\inf S$ exists. $\inf S = -\sup \bar{S}$.

Pf of Claim:

Check: $-\sup \bar{S}$ is a lower bd for S

(Ex:) This is the same by reversing the arguments of the claim above.

Check: $-\sup \bar{S}$ is the greatest lower bd. for S

Let $\varepsilon > 0$ be fixed but arbitrary.

(Want to show: $\exists\, s' \in S$ s.t. $-\sup \bar{S} + \varepsilon > s'$) — $(*)$

By ② of supremum for $\bar{S}$,

$$\sup \bar{S} - \varepsilon < \bar{s}' \quad \text{for some } \bar{s}' \in \bar{S}$$

By defn, we write $\bar{s}' = -s'$ for some $s' \in S$

So. $\sup \bar{S} - \varepsilon < -s' \Rightarrow -\sup \bar{S} + \varepsilon > s'$ for some $s' \in S$

which is $(*)$ ▫

Archimedean Property: $\mathbb{N}$ is NOT bdd above.

Pf: Suppose NOT, i.e. $\mathbb{N}$ is bdd above.

By Completeness Property, $\sup \mathbb{N} =: u \in \mathbb{R}$ exists.

So, $u - 1 < n'$ for some $n' \in \mathbb{N}$.

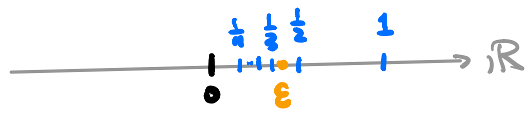
$\Rightarrow \quad u < n' + 1 \in \mathbb{N}$

$\Rightarrow \quad u$ is NOT an upper bd for $\mathbb{N}$ ← Contradiction!

$\blacksquare$

Corollaries:

(i) $\inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = 0$



(ii) $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \varepsilon$

(iii) $\forall y > 0, \exists! \, n \in \mathbb{N}$ s.t. $n - 1 < y \leq n$

unique

Ex: Prove these!